Journal of Algebra **242,** 146–159 (2001)
doi:10.1006/jabr.2001.8784, available online at http://www.idealibrary.com on IDEAL®

Costar Modules

Robert R. Colby[1] and Kent R. Fuller

University of Iowa, Iowa City, Iowa 52242

metadata, citation and similar papers at core.ac.uk

Received February 10, 2000

1. INTRODUCTION

Both quasi-progenerators [13] and tilting modules over arbitrary rings [6, 17, 21] induce equivalences between certain categories of modules that are generalizations of Morita equivalence. In [20] Menini and Orsatti introduced a generalization of these modules that have come to be called $*$-modules. A module ${}_RV$ with endomorphism ring S is a $*$-module if it induces an equivalence

$$\mathrm{Hom}_R(V, -) : \mathrm{Gen}({}_RV) \leftrightarrows \mathrm{Cogen}({}_SW) : (V \otimes_S -)$$

when ${}_SW = \mathrm{Hom}_R(V, C)$ with ${}_RC$ an injective cogenerator.

Over the years, several authors ([16, 19, 22, 25], for example) investigated a dual notion of quasi-progenerators, called quasi-duality modules in [14, 15], and cotilting modules [2, 4, 9], dual to tilting modules, have been a central topic of recent investigations in module theory. Both types of modules induce generalizations of Morita duality. Here we consider a class of modules that contains the quasi-duality modules and the cotilting modules. They are, in a sense, dual to $*$-modules so we call them *costar* modules.

Given a bimodule ${}_SQ_R$ we denote both contravariant functors $\mathrm{Hom}_R(\_, {}_SQ_R)$ and $\mathrm{Hom}_S(\_, {}_SQ_R)$ by Δ, and we let δ represent both the evaluation maps

$$\delta : 1_{\text{Mod-}R} \to \Delta^2 \quad \text{and} \quad \delta : 1_{S\text{-Mod}} \to \Delta^2.$$

[1] Colby expresses his gratitude for the hospitality of the University of Iowa where he is an independent scholar.

0021-8693/01 $35.00
Copyright © 2001 by Academic Press
All rights of reproduction in any form reserved.

A module M is (Q)-*reflexive* (*torsionless*) if δ_M is an isomorphism (a monomorphism). Denoting the classes of Q-reflexive right-R and left-S modules by $\operatorname{refl}(Q_R)$ and $\operatorname{refl}({}_SQ)$, respectively, it is always the case that ${}_SQ_R$ induces a duality

$$\Delta : \operatorname{refl}(Q_R) \leftrightarrows \operatorname{refl}({}_SQ) : \Delta.$$

A *costar module* is a module Q_R with endomorphism ring $S = \operatorname{End}(Q_R)$ such that the ${}_SQ_R$-duals induce a duality

$$\Delta : \text{fgd-tl}(Q_R) \leftrightarrows \text{fg-tl}({}_SQ) : \Delta$$

between the class of torsionless right R-modules whose Q-duals are finitely generated over S and the class of finitely generated torsionless left S-modules. In other words, every torsionless right R-module whose Q-dual is finitely generated is reflexive, and every finitely generated torsionless left S-module is reflexive.

Colpi presented several characterizations of $*$-modules in [7, Theorem 4.1 and Proposition 4.3]. In Section 2 we show that dual versions of these characterizations are valid for costar modules, and we show that the functors Δ preserve exactness of short exact sequences in fgd-tl(Q_R) and fg-tl$({}_SQ)$; cf. [7, Corollary 4.2; 10, Proposition 1.2]. In Section 3 we demonstrate, via a result of D'Este and Happel [12], that over a finite dimensional k-algebra, finitely generated costar modules are actually the k-duals of $*$-modules. We examine the relationship of costar modules to quasi-duality modules and cotilting modules in Section 4. As promised, both of the latter are costar modules, as are the finitely cotilting modules introduced by Angeleri Hügel [18]. We also determine when costar modules are quasi-duality modules. There is a natural strengthening of costar modules that we consider in Section 5. The class of these modules contains all cotilting modules and all finitely generated costar modules over a finite dimensional algebra. It also contains the bimodules that induce a generalized Morita duality in the sense of [4, 5]; we do not know if it contains all quasi-duality modules.

A module ${}_RM$ belongs to $\operatorname{cogen}({}_RU)$ ($\operatorname{Cogen}({}_RU)$) if M embeds in a finite (arbitrary) direct product of copies of U. The dual notions are $\operatorname{gen}({}_RU)$ and $\operatorname{Gen}({}_RU)$. If there is an embedding of M in a finite (arbitrary) direct product of copies of U that has a cokernel in $\operatorname{cogen}({}_RU)$ ($\operatorname{Cogen}({}_RU)$), we write $M \in \operatorname{copres}({}_RU)$ ($M \in \operatorname{Copres}({}_RU)$). These are the dual notions of finitely presented by and presented by U. For other terminology we defer to [1].

2. COSTAR MODULES

In order to obtain our dual version of Colpi's characterizations of $*$-modules, we begin with a series of lemmas regarding the Δ-functors induced by an arbitrary bimodule ${}_SQ_R$. Results similar to at least the first three of these have been applied previously in the literature; we include them for reference purposes. We denote the kernel of δ_M by $\mathrm{rej}_Q(M)$, so $M \in \mathrm{Cogen}(Q)$ if and only if $\mathrm{rej}_Q(M) = 0$.

LEMMA 2.1. *If M is a module, then δ_M is an epimorphism if and only if $M/\mathrm{rej}_Q(M)$ is reflexive.*

Proof. Let $M \xrightarrow{\pi} M/\mathrm{rej}_Q(M)$ be the projection. Then $\Delta(\pi)$ is an isomorphism by definition of reject, so $\Delta^2(\pi)$ is also an isomorphism and the commutativity of the diagram

$$\begin{array}{ccccc} M & \xrightarrow{\pi} & M/\mathrm{rej}_Q(M) & \longrightarrow 0 \\ {\scriptstyle \delta_M}\downarrow & & {\scriptstyle \delta_{M/\mathrm{rej}_Q(M)}}\downarrow & \\ \Delta^2 M & \xrightarrow{\Delta^2\pi} & \Delta^2(M/\mathrm{rej}_Q(M)) & \end{array}$$

completes the proof. ∎

LEMMA 2.2. *Suppose that $0 \to M \xrightarrow{f} X \to L \to 0$ is exact where X is reflexive and Δf is an epimorphism. Then M is reflexive if and only if L is torsionless.*

Proof. Applying Δ we obtain the exact sequence $0 \to \Delta L \to \Delta X \xrightarrow{\Delta f} \Delta M \to 0$ and then the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow & M & \xrightarrow{f} & X & \longrightarrow & L & \longrightarrow 0 \\ & {\scriptstyle \delta_M}\downarrow & & {\scriptstyle \delta_X}\downarrow & & {\scriptstyle \delta_L}\downarrow & \\ 0 \longrightarrow & \Delta^2 M & \xrightarrow{\Delta^2 f} & \Delta^2 X & \longrightarrow & \Delta^2 L & \end{array}$$

Since δ_X is an isomorphism the lemma follows from the snake lemma. ∎

To the best of our knowledge, the first version of the next lemma was presented by Bongartz [3].

LEMMA 2.3. *Suppose that M in* Mod-R *is such that $\Delta(M)$ in* S-Mod *has a set of generators with cardinality A. Then there is a monomorphism $M/\mathrm{rej}_Q(M) \xrightarrow{f} Q^A$ such that $\Delta(f)$ is an epimorphism. In particular*

$$\text{fgd-tl}(Q_R) \subseteq \mathrm{cogen}(Q_R).$$

Proof. Compare [3, proof of Proposition 1.4]. ∎

Since Q-duals are always Q-torsionless we always have

$$\Delta : \text{fgd-tl}(Q_R) \to \text{fg-tl}({}_SQ).$$

On the other hand, if ${}_SN$ is finitely generated, then applying Δ to the appropriate ${}_SS$-presentation of N we obtain a Q_R copresentation of $M_R = \Delta N$ of the form

$$0 \to M_R \to Q^n \to Q^A$$

with $n \in \mathbf{N}$. When M_R has such a copresentation, we say that M is *semifinitely copresented* by Q_R, and we write $M \in \text{sf-cp}(Q_R)$. Thus we have

$$\Delta : \text{fg-tl}({}_SQ) \to \text{sf-cp}(Q_R).$$

Now it becomes apparent that in order to have the costar duality we seek, it is necessary that $\text{fgd-tl}(Q_R) \subseteq \text{sf-cp}(Q_R)$, and also, if ${}_SQ$ is faithful (i.e., if ${}_SS \in \text{fg-tl}({}_SQ)$) it is necessary that $S \cong \text{End}(Q_R)$ and hence that Q_R and ${}_SS$ are both Q-reflexive.

LEMMA 2.4. *Assume that $S = \text{End}(Q_R)$. Then $M_R \in \text{fgd-tl}(Q_R)$ if and only if there is a monomorphism $M \xrightarrow{f} Q^n$ with $\Delta(f)$ an epimorphism.*

Proof. If $M \in \text{fgd-tl}(Q_R)$ Lemma 2.3 gives the required monomorphism. Conversely, the exact sequence $S^n \cong \Delta Q^n \xrightarrow{\Delta f} \Delta M \to 0$ shows that $\Delta M \in S\text{-mod}$. ∎

Now we come to the lemma that is a central element in several of our main results.

LEMMA 2.5. *Suppose that $0 \to M \xrightarrow{f} X \to L \to 0$ is exact in* Mod-R. *Denote the image of Δf by I and let $\Delta X \xrightarrow{\pi} I \xrightarrow{j} \Delta M$ be the factorization of Δf through I with j the inclusion. Let $\alpha = \Delta j \circ \delta_M$ so that the diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{f} & X & \longrightarrow & L & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\delta_X}\big\downarrow & & {\scriptstyle\delta_L}\big\downarrow & & \\
0 & \longrightarrow & \Delta I & \xrightarrow{\Delta\pi} & \Delta^2 X & \longrightarrow & \Delta^2 L & &
\end{array}$$

is commutative with exact rows. Then $j = \Delta\alpha \circ \delta_I$. Consequently,

(i) *if X is reflexive and L is torsionless then Δf is epic if and only if I is reflexive.*

Moreover, when $S = \mathrm{End}(Q_R)$, *then*

(ii) *if* $X = Q^n$, $L \in \mathrm{Cogen}(Q_R)$, *and* $\text{fg-tl}({}_SQ) \subseteq \mathrm{refl}({}_SQ)$ *then* Δf *is epic*; *and*

(iii) *if* $\text{fg-tl}({}_SQ) \subseteq \mathrm{refl}({}_SQ)$, *then* $\text{sf-cp}(Q_R) \subseteq \text{fgd-tl}(Q_R)$.

Proof. By the adjointness of the Δ functors we have

$$\Delta(\delta_M) \circ \delta_{\Delta M} = 1_{\Delta M}$$

and since δ is a natural transformation

$$\delta_{\Delta M} \circ j = \Delta^2 j \circ \delta_I.$$

Also $\Delta\alpha = \Delta(\Delta j \circ \delta_M) = \Delta(\delta_M) \circ \Delta^2(j)$, so

$$j = 1_{\Delta M} \circ j = \Delta(\delta_M) \circ \delta_{\Delta M} \circ j = \Delta(\delta_M) \circ \Delta^2 j \circ \delta_I = \Delta\alpha \circ \delta_I.$$

For (i), if X is reflexive and L is torsionless, then α is an isomorphism by the five lemma so $\Delta\alpha$ is also an isomorphism. Hence Δf is epic if and only if j is an isomorphism if and only if δ_I is an isomorphism. (ii) follows from (i) since $\Delta(Q^n) = S^n$ implies $I \in \text{fg-tl}({}_SQ)$. For (iii), if $M \in \text{sf-cp}(Q_R)$ there is an exact sequence $0 \to M \xrightarrow{f} Q^n \to L \to 0$ with $L \in \mathrm{Cogen}(Q_R)$ and Δf is epic by (ii). Hence $M \in \text{fgd-tl}(Q_R)$ by Lemma 2.4. ∎

According to our next result, if ${}_SQ$ is faithful, the finitely generated torsionless left S-modules are reflexive if and only if they are Δ-dual to the semifinitely copresented right R-modules.

PROPOSITION 2.6. *Assume that* $S = \mathrm{End}(Q_R)$. *The following are equivalent.*

(a) $\Delta : \text{sf-cp}(Q_R) \leftrightarrows \text{fg-tl}({}_SQ) : \Delta$ *is a duality.*

(b) $\text{fg-tl}({}_SQ) \subseteq \mathrm{refl}({}_SQ)$.

(c) *If* $0 \to M \xrightarrow{f} Q^n \to L \to 0$ *is exact with* $L \in \mathrm{Cogen}(Q_R)$, *then* $\Delta(f)$ *is an epimorphism.*

Proof. (a) $\Rightarrow$ (b) is obvious and (b) $\Rightarrow$ (c) follows from Lemma 2.5(ii).

(c) $\Rightarrow$ (a). By Lemma 2.4, $\text{sf-cp}(Q_R) \subseteq \text{fgd-tl}(Q_R)$ so $\Delta : \text{sf-cp}(Q_R) \to \text{fg-tl}({}_SQ)$. Thus we have $\Delta : \text{sf-cp}(Q_R) \leftrightarrows \text{fg-tl}({}_SQ) : \Delta$. If $M \in \text{sf-cp}(Q_R)$ then applying (c) and Lemma 2.2 we conclude that M is reflexive. Suppose that ${}_SN \in \text{fg-tl}({}_SQ)$. Then there is an exact sequence $0 \to K \to S^m \xrightarrow{g} {}_SN \to 0$ and, applying Δ we obtain an exact sequence $0 \to \Delta(N) \xrightarrow{\Delta g} \Delta(S^m) \to L \to 0$ where $\Delta(S^m) \cong Q^m$ and $L \in \mathrm{Cogen}(Q_R)$ since L embeds in $\Delta K \in$

$\mathrm{Cogen}(Q_R)$. By (c), $\Delta^2 g$ is epic and we obtain that N is reflexive from the following commutative diagram with exact rows, since S is reflexive.

$$\begin{array}{ccccc} S^m & \xrightarrow{g} & N & \longrightarrow & 0 \\ {\scriptstyle \delta_S}\downarrow & & {\scriptstyle \delta_N}\downarrow & & \\ \Delta^2 S^m & \xrightarrow{\Delta^2 g} & \Delta^2 N & \longrightarrow & 0 \end{array}$$

∎

Now we have the promised characterizations of costar modules.

THEOREM 2.7. *Assume that* $S = \mathrm{End}(Q_R)$. *The following are equivalent.*

(a) $\Delta : \text{fgd-tl}(Q_R) \leftrightarrows \text{fg-tl}({}_SQ) : \Delta$ *is a duality. That is,* Q_R *is a costar module.*

(b) $\Delta : \text{sf-cp}(Q_R) \leftrightarrows \text{fg-tl}({}_SQ) : \Delta$ *is a duality and* $\text{fgd-tl}(Q_R) = \text{sf-cp}(Q_R)$.

(c) δ_M *is an epimorphism if* $\Delta M \in S\text{-mod}$ *and* δ_N *is an epimorphism if* $N \in S\text{-mod}$.

(d) $\text{fgd-tl}(Q_R) \subseteq \text{sf-cp}(Q_R)$ *and if* $0 \to M \xrightarrow{f} Q^n \to L \to 0$ *is exact with* $L \in \mathrm{Cogen}(Q_R)$, *then* $\Delta(f)$ *is an epimorphism.*

(e) *If* $0 \to M \xrightarrow{f} Q^n \to L \to 0$ *is exact then* $L \in \mathrm{Cogen}(Q_R)$ *if and only if* $\Delta(f)$ *is an epimorphism.*

Proof. (b) $\Leftrightarrow$ (d) by Proposition 2.6 and Lemma 2.4, and (a) $\Rightarrow$ (c) by Lemma 2.1. By Lemma 2.1, (c) $\Rightarrow$ (a) will follow if we show that $\Delta : \text{fg-tl}({}_SQ) \to \text{fgd-tl}(Q_R)$. By (c), $\text{fg-tl}({}_SQ) \subseteq \text{refl}\,{}_SQ$. Hence $\text{sf-cp}(Q_R) \subseteq \text{fgd-tl}(Q_R)$ by Lemma 2.5(iii) and (a) follows. Assuming (a), $\text{fgd-tl}(Q_R) \subseteq \text{refl}(Q_R)$ so $\text{fgd-tl}(Q_R) \subseteq \text{sf-cp}(Q_R)$. Hence $\text{fgd-tl}(Q_R) = \text{sf-cp}(Q_R)$ by Lemma 2.5(iii) and we have (b). Thus, since (b) $\Rightarrow$ (a) is obvious, (a) through (d) are equivalent.

(d) $\Rightarrow$ (e). Suppose that $0 \to M \xrightarrow{f} Q^n \to L \to 0$ is exact with Δf epic. Then $M \in \text{fgd-tl}(Q_R)$ by Lemma 2.4 so M is reflexive since (d) $\Rightarrow$ (a). Since Q^n is reflexive by hypothesis, $L \in \mathrm{Cogen}(Q_R)$ by Lemma 2.2.

(e) $\Rightarrow$ (b). By Proposition 2.6 it suffices to show that $\text{fgd-tl}(Q_R) = \text{sf-cp}(Q_R)$. If $M \in \text{fgd-tl}(Q_R)$ then by Lemma 2.4 there is an exact sequence $0 \to M \xrightarrow{f} Q^n \to L \to 0$ with Δf epic. Hence (e) implies that $M \in \text{sf-cp}(Q_R)$. If $M \in \text{sf-cp}(Q_R)$, then $M \in \text{fgd-tl}(Q_R)$ by Lemma 2.4. ∎

Before proceeding further, we note that the modules inducing a duality between $\text{sf-cp}(Q_R)$ and $\text{fg-tl}({}_SQ)$ need not be costar modules. Indeed, any injective module satisfies condition (c) of Proposition 2.6 but not necessarily condition (e) of Theorem 2.7.

Condition (e) of Theorem 2.7 can be strengthened a bit, and this leads to some exactness properties of costar modules.

PROPOSITION 2.8. *Suppose that Q_R is a costar module. If $0 \to M \xrightarrow{f} X \to L \to 0$ is exact with $X \in$ fgd-tl(Q_R), then Δf is epic if and only if $L \in$ Cogen(Q_R). In this case, $M \in$ fgd-tl(Q_R).*

Proof. Assume that $0 \to M \xrightarrow{f} X \xrightarrow{\eta} L \to 0$ is exact. First assume that Δf is epic. Since $\Delta X \in$ fg-tl$({}_SQ)$, ΔM is finitely generated so by Theorem 2.7(c), δ_M is epic and hence is an isomorphism. Thus $L \in$ Cogen(Q_R) by Lemma 2.2. Conversely, assume that $L \in$ Cogen(Q_R). Consider the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "f"] & X \arrow[r, "\eta"] \arrow[d, "i"] & L \arrow[r] \arrow[d, "j"] & 0 \\
 & & Q^m \arrow[r, "\alpha"] \arrow[d, "g"] & Q^B & \\
 & & Q^A & &
\end{tikzcd}$$

where we have obtained α with $\alpha i = j\eta$ from the epimorphism (Proposition 2.6(c))

$$\mathrm{Hom}_R(Q^m, Q^B) \to \mathrm{Hom}_R(X, Q^B).$$

It is, then, straightforward that Im(if) = Ker $\alpha \cap$ Ker g. Since if is monic we obtain an exact sequence

$$0 \to M \xrightarrow{if} Q^m \to Q^A \times Q^B.$$

Thus $M \in$ sf-cp(Q_R) = fgd-tl(Q_R) and $\Delta(if) = \Delta(f) \circ \Delta(i)$ is epic by Proposition 2.6(c), and hence $\Delta(f)$ is also epic. ∎

Dual to [7, Corollary 4.2; 10, Proposition 1.2] we have

COROLLARY 2.9. *If Q_R is a costar module then the Δ's preserve exactness of short exact sequences of modules in both in* fgd-tl(Q_R) *and* fg-tl$({}_SQ)$.

Proof. The first assertion is immediate from Proposition 2.8. For the second, suppose that

$$0 \to K \xrightarrow{g} N \xrightarrow{f} L \to 0$$

is exact with K, N, L in fg-tl$({}_SQ)$; then in the exact sequence

$$0 \to \Delta L \xrightarrow{\Delta f} \Delta N \xrightarrow{\Delta g} \Delta K$$

$\Delta N \in$ fgd-tl(Q_R) is reflexive and $I = \operatorname{Im} \Delta g$ is torsionless. Thus by Proposition 2.8 the bottom row is exact in

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "g"] \arrow[d] & N \arrow[r, "f"] \arrow[d, "\delta_N"'] & L \arrow[r] \arrow[d, "\delta_L"'] & 0 \\
0 \arrow[r] & \Delta I \arrow[r] & \Delta^2 N \arrow[r, "\Delta^2 f"] & \Delta^2 L \arrow[r] & 0
\end{tikzcd}$$

so $\Delta I \cong K$ is finitely generated, and $I \in$ fgd-tl(Q_R) is reflexive. Thus Δg is epic by Lemma 2.5(i). ∎

3. COSTAR MODULES OVER FINITE DIMENSIONAL ALGEBRAS

According to [12] if R is a finite dimensional algebra over a field k, then any faithful $*$-module is a tilting module; and it follows from [18, Theorem 3.3 and Proposition 2.2] that a finitely generated module Q_R is a cotilting module if and only if its vector space dual, the left R-module $D(Q_R)$, is a tilting module. Employing these results, in this section we show that, for finitely generated R-modules, the notion of costar modules is, as one would hope, the dual to that of $*$-modules.

LEMMA 3.1. *If R is a finite dimensional algebra over a field k and $Q_R \in$ mod-R, then* fgd-tl(Q_R) = fg-tl(Q_R) = cogen(Q_R).

Proof. It is clear that fg-tl$(Q_R) \subseteq$ cogen(Q_R) and cogen$(Q_R) \subseteq$ fg-tl(Q_R). Also fg-tl$(Q_R) \subseteq$ fgd-tl(Q_R) since $\operatorname{Hom}_R(M, Q) \subseteq \operatorname{Hom}_k(M, Q)$, and fgd-tl$(Q_R) \subseteq$ fg-tl(Q_R) by Lemma 2.4. ∎

Dual to the case for faithful star modules we have

PROPOSITION 3.2. *If Q_R is a faithful finitely generated costar module over a finite dimensional algebra R, then Q_R is a cotilting module.*

Proof. Since Q_R is faithful, finitely generated projective R-modules are in cogen(Q_R) and are reflexive by Lemma 3.1. Let $M \in$ cogen(Q_R). By Lemma 3.1 and Theorem 2.7, M is reflexive and finitely generated. Let $0 \to K \xrightarrow{f} P \to M \to 0$ be exact with P finitely generated and projective. Then, denoting $\operatorname{Ext}^1(-, Q)$ by Γ, $0 \to \Delta M \to \Delta P \to \Delta K \to \Gamma M \to 0$ is exact and the image I of Δf is reflexive since $I \in$ fg-tl$({}_SQ)$ ($I \subseteq \Delta K$ and $\Delta P \in$ add$({}_SQ)$ are finitely generated). Hence Δf is epic by Lemma 2.5(i) and we obtain $\Gamma M = 0$. On the other hand, if $\Gamma M = 0$ and M is finitely generated then $0 \to \Delta M \to \Delta P \to \Delta K \to 0$ is exact and $K \in \mathrm{cogen}(P) \subseteq \mathrm{cogen}(Q_R)$ is reflexive, so M is torsionless by Lemma 2.2. Thus we have shown that

$$\mathrm{cogen}(Q_R) = (\mathrm{Ker}\,\Gamma) \cap (\text{mod-}R).$$

According to [18, Theorems 2.2 and 3.3], to complete the proof, we simply need to verify the three conditions

(i) $\mathrm{Ext}^1_R(Q, Q) = 0$;

(ii) inj. dim. $Q_R \leq 1$;

(iii) $\mathrm{Ker}\,\Gamma \cap \mathrm{Ker}\,\Delta \cap \text{mod-}R = 0$.

By the argument above, (i) is clear, and since submodules of finitely generated projectives are in cogen(Q_R) and hence in Ker Γ we obtain that $\mathrm{Ext}^2_R(M, Q) = 0$ for all finitely generated M. Hence inj. dim. $Q_R \leq 2$ by Baer's criterion. For (iii), if $M \in \text{mod-}R$ and $\Gamma M = 0$, then $M \in \mathrm{cogen}(Q_R)$, as noted above. Thus $\Gamma M = 0 = \Delta M$ implies $M = 0$. ▮

As promised, we now have

PROPOSITION 3.3. *A finitely generated module Q_R over a finite dimensional k-algebra R is a costar module if and only if ${}_RV = D(Q_R)$ is a $*$-module.*

Proof. Let $I = \mathrm{Ann}_R(V) = \mathrm{Ann}_R(Q)$ and let $\overline{R} = R/I$. Then, according to [12, Corollary 2], ${}_RV$ is a $*$-module if and only if ${}_{\overline{R}}V$ is a tilting module. But clearly if Q_R is a costar module, then so is $Q_{\overline{R}}$, and so by Proposition 3.2 ${}_{\overline{R}}V = D(Q_{\overline{R}})$ is a tilting module and ${}_RV = D(Q_R)$ is a $*$-module. Conversely, suppose that ${}_RV$ is a star module with $S = \mathrm{End}({}_RV)$, choose the cogenerator ${}_RC = D(R_R)$, and let ${}_SQ_R = D(V)$. Then, using Lemma 3.1, one can easily verify that the Δ-functors

$$\Delta = \mathrm{Hom}_R(-, Q) \cong \mathrm{Hom}_R(V, -) \circ D$$

and

$$\Delta = \mathrm{Hom}_S(-, Q) \cong D \circ (V \otimes_S -)$$

induce a duality between fgd-tl(Q_R) and fg-tl(${}_SQ$) = cogen($\mathrm{Hom}_R(V, C)$). ▮

4. RELATIONSHIPS WITH QUASI-DUALITY AND COTILTING

A module Q_R with $S = \mathrm{End}(Q_R)$ is a quasi-duality module if the ${}_SQ_R$-duals induce a duality

$$\Delta : \overline{\mathrm{gen}}(Q_R) \leftrightarrows \overline{\mathrm{gen}}({}_SS) : \Delta$$

between the smallest finitely closed subcategories of Mod-R and S-Mod that contain Q_R and ${}_SS$, respectively. As we noted earlier several authors investigated these modules. Among their characterizations: Q_R is a quasi-duality module if and only if ${}_SQ$ is an injective cogenerator and Q_R is a quasi-injective module which cogenerates all epimorphic images of Q_R. (See [16] or [14], for example.) It follows from this result that they are costar modules.

PROPOSITION 4.1. *Every quasi-injective R-module which cogenerates all its factors is a costar module.*

Proof. Since the hypothesis implies that Q cogenerates all factors of Q^n with $n \in \mathbf{N}$ (as can be seen using a proof similar to [13, Lemma 2.2]) this follows from Theorem 2.7(e). ▮

In fact we can pinpoint quasi-duality modules among the costar modules.

PROPOSITION 4.2. *Let $S = \mathrm{End}(Q_R)$. Then Q_R is a quasi-duality module if and only if Q_R is a costar module such that every factor of ${}_SS$ is torsionless and every factor of Q_R is reflexive.*

Proof. If Q_R is a quasi-duality module, then Q_R is a costar module, as we just observed in Proposition 4.1, and the defining duality shows that it satisfies the remaining two conditions.

Conversely, assume that Q_R is a costar module satisfying the stated conditions. Since every factor of Q_R is reflexive and hence torsionless, given any exact sequence $0 \to M \xrightarrow{f} Q \to L \to 0$ in Mod-R, Δf is epic by Theorem 2.7(e) so Q_R is quasi-injective. To see that ${}_SQ$ is injective, suppose that $0 \to K \xrightarrow{g} S \to N \to 0$ is exact in S-Mod. Then N is torsionless, S is reflexive, and $I = \mathrm{Im}\ \Delta g$ is reflexive since $\Delta S \cong Q_R$. Thus Δg is epic by 2.5(i). Also by hypothesis, ${}_SQ$ cogenerates all simple S-modules so ${}_SQ$ is an injective cogenerator. ▮

The notion of a cotilting module over an arbitrary ring R was introduced in [9, 11] as a module Q_R such that

$$\mathrm{Ker}\ \Gamma = \mathrm{Cogen}(Q_R)$$

where $\Gamma = \mathrm{Ext}^1_R(-, Q)$. As promised, these too are costar modules.

PROPOSITION 4.3. *Suppose that Q_R is a cotilting module and $S =$ $\mathrm{End}(Q_R)$. If $0 \to M \xrightarrow{f} X \to L \to 0$ is exact with X reflexive, then Δf is epic if and only if $L \in \mathrm{Cogen}(Q_R)$. Hence Q_R is a costar module.*

Proof. If $0 \to M \xrightarrow{f} X \to L \to 0$ is exact with X torsionless, then since $\mathrm{Ker}\ \Gamma = \mathrm{Cogen}(Q_R)$ the sequence $\Delta X \xrightarrow{\Delta f} \Delta M \to \Gamma L \to \Gamma X = 0$ is exact so the result follows. Then Q_R is a costar module by Theorem 2.7(e). ▮

In [8] the authors found an example of a faithful $*$-module that was neither a tilting module nor a quasi-progenerator. Since a non-finitely generated **Z**-torsionless abelian group must contain free direct summands of arbitrarily large finite rank, fgd-tl($\mathbf{Z}_{\mathbf{Z}}$) and fg-tl($_{\mathbf{Z}}\mathbf{Z}$) are both just the finitely generated free abelian groups. Thus, it is a lot simpler to find such an example for costar modules.

Example 4.4. **Z** is a faithful costar module that is neither a cotilting module nor a quasi-duality module.

The so called *finitely cotilting* modules recently introduced by Hügel in [18] are also contained in the class of costar modules.

PROPOSITION 4.5. *Suppose that Q_R is finitely cotilting, then Q_R is a costar module.*

Proof. A finitely cotilting module Q_R is finitely generated, has $\Gamma(Q_R) = 0$ and $\mathrm{Cogen}(Q_R) \cap \text{mod-}R = \mathrm{Ker}\,\Gamma \cap \text{mod-}R$ [18, Proposition 1.1(3)]. Hence it is routine to verify (e) of Theorem 2.7. ∎

Even more recently, Tonolo [24] investigated a class of bimodules that contains the two-sided finitely cotilting bimodules. He called them *FWC-bimodules*. They may not be costar modules, but we do have

PROPOSITION 4.6. *If $_SQ_R$ is a FWC-bimodule with $S = \mathrm{End}(Q_R)$ then Q_R satisfies the conditions of Proposition* 2.6.

Proof. If $0 \to M \xrightarrow{f} Q^n \to L \to 0$ is exact with $L \in \mathrm{Cogen}(Q_R)$ then $\Gamma L = 0$ by definition of a FWC-module (see [24]), so Δf is an epimorphism. ∎

5. STRONG COSTAR MODULES

According to [4, 5], a faithfully balanced bimodule $_SQ_R$ induces a *generalized Morita duality* (GMD) if refl(Q_R) and refl($_SQ$) are closed under submodules and extensions. To confirm that cotilting modules are costar modules in Proposition 4.3 we actually proved that they satisfy a condition ((a) in the following theorem) that appears to be stronger than condition (e) of Theorem 2.7. This condition is equivalent to a pair of conditions that are reminiscent of properties of the categories that formed the framework for GMDs.

THEOREM 5.1. *For a bimodule $_SQ_R$, the following are equivalent.*

(a) *If $0 \to M \xrightarrow{f} X \to L \to 0$ is exact in* Mod-R *and X is reflexive then L is torsionless if and only if Δf is an epimorphism.*

(b) (i) *δ_N is epic if $_SN$ is an epimorph of a reflexive S-module, and*

(ii) δ_M *is epic if* $M \in \text{Mod-}R$ *and* ΔM *is a reflexive* S*-module*.

Proof. (b) ⇒ (a). Suppose $0 \to M \xrightarrow{f} X \to L \to 0$ is exact in Mod-R with X reflexive. Then ΔX is reflexive. Hence, if L is torsionless, by (b)(i) we can apply Lemma 2.5(i) to conclude that Δf is epic. On the other hand, if Δf is epic, i.e., the sequence $\Delta X \xrightarrow{\Delta f} \Delta M \to 0$ is exact, then ΔM is reflexive by (b)(i) and then δ_M is epic by (b)(ii). Hence M is reflexive so we obtain that L is torsionless by Lemma 2.2.

(a) ⇒ (b). Suppose that $0 \to K \to Y \xrightarrow{\pi} N \to 0$ is exact in S-Mod with Y reflexive. Then ΔY is reflexive and we obtain an exact sequence $0 \to \Delta N \xrightarrow{\Delta\pi} \Delta Y \to L \to 0$ where $L \subseteq \Delta K$ is torsionless. By (a) $\Delta^2\pi$ is epic so δ_N is epic since we have the commutative diagram with exact rows

$$\begin{array}{ccccc} Y & \xrightarrow{\pi} & N & \longrightarrow & 0 \\ \cong\big\downarrow \delta_Y & & \big\downarrow \delta_N & & \\ \Delta^2 Y & \xrightarrow{\Delta^2\pi} & \Delta^2 N & \longrightarrow & 0 \end{array}$$

Thus we have proved (i). Now assume that M_R is torsionless and ΔM is reflexive. In the exact sequence $0 \to M \xrightarrow{\delta_M} \Delta^2 M \to L \to 0$, $\Delta(\delta_M)$ is epic since $\Delta(\delta_M) \circ \delta_{\Delta M} = 1_{\Delta M}$. By (a) L is torsionless and using again that $\Delta(\delta_M)$ is epic we obtain that M is reflexive by Lemma 2.2. Then (ii) follows for arbitrary M_R by Lemma 2.1. ▮

We call a module Q_R with $S = \text{End}(Q_R)$ which satisfies the conditions of Theorem 5.1 a *strong costar module* and remark again that cotilting modules are such. So are the bimodules that induce GMDs. (It is still an open question whether cotilting modules over a finite dimensional algebra induce GMDs.)

PROPOSITION 5.2. *If submodules of reflexive modules are reflexive*, *then* Q_R *is a strong costar module*.

Proof. We verify (a) of Theorem 5.1. Suppose that $0 \to M \xrightarrow{f} X \to L \to 0$ is exact with X reflexive. Then M is reflexive by hypothesis. If Δf is epic, then L is torsionless by Lemma 2.2. Suppose that L is torsionless. In Lemma 2.5, I is reflexive since $I \subseteq \Delta M$. Hence Δf is epic by Lemma 2.5(i). ▮

Proposition 5.2 shows that a large class of quasi-duality modules are strong costar modules (see [15, Proposition 3.2]), but we do not know if this is true of all quasi-duality modules.

A strong costar module that is balanced satisfies the conditions of Colby's definition of cotilting modules over Noetherian rings [4, p. 1718]:

THEOREM 5.3. *Suppose that Q_R is a strong costar module and $R \in$ refl(Q_R). Then*

(i) $\mathrm{Ext}_R^2(M, Q) = 0$ *for all finitely presented R-modules.*

(ii) $\mathrm{Cogen}(Q_R) \cap \text{mod-}R = \mathrm{Ker}\,\Gamma \cap \text{mod-}R$.

(iii) $\mathrm{Ker}\,\Delta \cap \mathrm{Ker}\,\Gamma \cap \text{mod-}R = 0$.

Proof. (ii). Let $M \in \text{mod-}R$ and let $0 \to K \xrightarrow{f} P \to M \to 0$ be exact with P finitely generated and projective. Then P is reflexive by hypothesis. Consider the induced exact sequence

$$0 \to \Delta M \to \Delta P \xrightarrow{\Delta f} \Delta K \to \Gamma M \to 0.$$

Here ΔP is reflexive, and $I = \mathrm{Im}\ \Delta f$ is torsionless and hence reflexive by Theorem 5.1(b)(i). Thus by Lemma 2.5(i), if $M \in \mathrm{Cogen}(Q_R)$, then Δf is epic, so $\Gamma M = 0$. On the other hand, if $M \in \mathrm{Ker}\,\Gamma$ then Δf is epic, $\Delta K = I$ is reflexive, and since K is torsionless, K is reflexive by Theorem 5.1(b)(ii). Hence M is torsionless by Lemma 2.2.

(i) If M is finitely presented then, in the first exact sequence of the proof of (ii) above, $K \in \mathrm{Cogen}(Q_R) \cap \text{mod-}R$ so $\Gamma K = 0$ by (ii). (i) follows from the exactness of the sequence $0 = \Gamma K \to \mathrm{Ext}_R^2(M, Q) \to 0$.

(iii) If $\Gamma M = \Delta M = 0$ where $M \in \text{mod-}R$, then M is torsionless by (ii) and hence is zero. ∎

Regarding finite dimensional algebras, we conclude with

PROPOSITION 5.4. *If R is a finite dimensional algebra, then any finitely generated costar module over R is a strong costar module.*

Proof. Let Q_R be finitely generated and let $\overline{R} = R/\mathrm{Ann}_R(Q)$. Then it follows from (e) of Theorem 2.7 ((a) of Theorem 5.1) that Q_R is a costar (strong costar) module if and only if so is $Q_{\overline{R}}$. Thus if Q_R is a costar module, then $Q_{\overline{R}}$ is a cotilting module by Proposition 3.2 and a strong costar module by Proposition 4.3, and thus so is Q_R. ∎

REFERENCES

1. F. W. Anderson and K. R. Fuller, "Rings and Categories of Modules," 2nd ed., Springer-Verlag, New York/Heidelberg/Berlin, 1992.
2. I. Assem, "Tilting Theory—An Introduction," Topics in Algebra, Vol. 26, pp. 127–180, Banach Center Publications, 1990.
3. K. Bongartz, "Tilted Algebras," Lecture Notes in Mathematics, Vol. 903, pp. 26–38, Springer-Verlag, New York/Heidelberg/Berlin, 1981.

4. R. R. Colby, A generalization of Morita duality and the tilting theorem, *Comm. Alg.* **17**(7) (1989), 1709–1722.
5. R. R. Colby, A cotilting theorem for rings, *in* "Methods in Module Theory," pp. 33–37, Dekker, New York, 1993.
6. R. R. Colby and K. R. Fuller, Tilting, cotilting, and serially tilted rings, *Comm. Algebra* **18** (1990), 1585–1615.
7. R. Colpi, Some remarks on equivalences between categories of modules, *Comm. Algebra* **18** (1990), 1935–1951.
8. R. Colpi and G. D'Este, Equivalences represented by faithful non-tilting $*$-modules, *Canad. Math. Soc. Conf. Proc.* **24** (1998), 103–110.
9. R. Colpi, G. D'Este, and A. Tonolo, Quasi-tilting modules and counter equivalences, *J. Algebra* **191** (1997), 461–494.
10. R. Colpi and C. Menini, On the structure of $*$-modules, *J. Algebra* **158** (1993), 400–419.
11. R. Colpi, A. Tonolo, and J. Trlifaj, Partial cotilting modules and the lattices induced by them, *Comm. Algebra* **25** (1997), 3225–3237.
12. G. D'Este and D. Happel, Representable equivalences are represented by tilting modules, *Rend. Sem. Mat. Univ. Padova* **83** (1990), 77–80.
13. K. Fuller, Density and equivalence, *J. Algebra* **29** (1974), 528–550.
14. K. R. Fuller, A note on quasi-duality, *in* "Abelian Groups, Module Theory, and Topology" (D. Dikranjan and L. Salce, Eds.), Lecture Notes in Pure and Applied Mathematics, Vol. 201, Dekker, New York, 1998.
15. K. R. Fuller and W. Xue, On quasi-duality modules, *Comm. Algebra* **28** (2000), 1919–1937.
16. J. L. Gómez Pardo, P. A. Guil Asensio, and R. Wisbauer, Morita dualities induced by the m-dual functors, *Comm. Algebra* **22** (1994), 5903–5934.
17. D. Happel and C. M. Ringel, Tilted algebras, *Trans. Amer. Math. Soc.* **274** (1982), 399–443.
18. L. Angeleri Hügel, Finitely cotilting modules, *Comm. Algebra* **28** (2000), 2147–2172.
19. C. Menini and A. Orsatti, Good dualities and strongly quasi-injective modules, *Ann. Mat. Pure Appl.* (*4*) **127** (1981), 187–230.
20. C. Menini and A. Orsatti, Representable equivalences between categories of modules and applications, *Rend. Sem. Mat. Univ. Padova* **82** (1989), 203–231.
21. Y. Miyashita, Tilting modules of finite projective dimension, *Math. Z.* **193** (1986), 113–146.
22. F. L. Sandomierski, Linearly compact modules and local Morita duality, *in* "Ring Theory" (R. Gordon, Ed.), pp. 333–346, Academic Press, New York, 1972.
23. H. Tachikawa, "Quasi Frobenius Rings and Generalizations," Lecture Notes in Mathematics, Vol. 351, Springer-Verlag, Berlin/Heidelberg/New York, 1973.
24. A. Tonolo, On a finitistic cotilting-type duality, *Comm. Algebra*, to appear.
25. J. M. Zelmanowitz and W. Jansen, Duality modules and Morita duality, *J. Algebra* **129** (1989), 257–277.